Design Involving Several Latin Squares

Kristian Nestor

March 2022

The $p \times p$ Latin square contains only p observations for each treatment. To obtain more replications, the experimenter may use several squares, say n. It is immaterial whether the squares used are the same or different. The appropriate model is

$$y_{ijkh} = \mu + \rho_h + a_{i(h)} + \tau_j + b_{k(h)} + (\tau\rho)_{jh} + \epsilon_{ijkh} \begin{cases} i = 1, 2, ..., p \\ j = 1, 2, ..., p \\ k = 1, 2, ..., p \\ h = 1, 2, ..., n \end{cases}$$

where y_{ijkh} is the observation on treatment j in row i and column k of the $h-th$ square, ρ_h is the effect of the $h-th$ square, and $(\tau\rho)_{jh}$ is the interaction between treatments and squares. [**See Cochran and Cox (1957), John (1971)**]

(a) Normal Equations

Assume that the appropriate side conditions on the parameters are

$$\sum_h \hat{\rho_h} = 0, \sum_i \hat{a}_{i(h)} = 0, \sum_k \hat{b}_{k(h)} = 0, \sum_j \hat{\tau}_j = 0, \sum_j \hat{(\tau\rho)}_{jh} = 0, for\ each\ h\ and\ \sum_h \hat{(\tau\rho)}_{jh} = 0, for\ each\ j$$

The normal equations for this model are

Greant Mean Effect

$$\mu : np^2\hat{\mu} + \sum_i \sum_k \sum_{j(i,k)} \left(\sum_h \hat{\rho}_h \right) + \sum_h \sum_k \sum_{j(i,k)} \left(\sum_i \hat{a}_{i(h)} \right) + \sum_h \sum_k \sum_i \left(\sum_{j(i,k)} \hat{\tau}_j \right)$$

$$+ \sum_h \sum_{j(i,k)} \sum_i \left(\sum_k \hat{b}_{k(h)} \right) + \sum_i \sum_k \sum_{j(i,k)} \left(\sum_h \hat{(\tau\rho)}_{jh} \right) = y_{....}$$

$$\implies \hat{\mu} = \bar{y}_{....}$$

h-th Latin Square Effect

$$\rho_h : p^2\hat{\mu} + \sum_i \sum_k \sum_{j(i,k)} \hat{\rho}_h + \sum_k \sum_{j(i,k)} \left(\sum_i \hat{a}_{i(h)} \right) + \sum_k \sum_i \left(\sum_{j(i,k)} \hat{\tau}_j \right)$$

$$+ \sum_{j(i,k)} \sum_i \left(\sum_k \hat{b}_{k(h)} \right) + \sum_i \sum_k \sum_{j(i,k)} \hat{(\tau\rho)}_{jh} = y_{...h}\ for\ each\ h,\ h = 1, 2, ..., n$$

$$\implies \hat{\rho}_h = \bar{y}_{...h} - \hat{\mu}$$

$$\implies \hat{\rho}_h = \bar{y}_{...h} - \bar{y}_{....},\ h = 1, 2, ..., n$$

i-th Row Effect in the k-th Latin Square

$$a_{i(h)} : p\hat{\mu} + \sum_k \sum_j \hat{\rho}_h + p\hat{a}_{i(h)} + \sum_k \sum_j \hat{\tau}_j + \sum_j \sum_k \hat{b}_{k(h)} + \sum_k \sum_j \hat{(\tau\rho)}_{jh} = y_{i..h}, \ for \ each \ i = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{a}_{i(h)} = \bar{y}_{i..h} - \hat{\mu} - \hat{\rho}_h, \quad i = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{a}_{i(h)} = (\bar{y}_{i..h} - \bar{y}_{....}) - (\bar{y}_{...h} - \bar{y}_{....}), \quad i = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{a}_{i(h)} = \bar{y}_{i..h} - \bar{y}_{...h}, \quad i = 1, .., p \ and \ h = 1, .., n$$

j-th Treatment Effect

$$\tau_j : pn\hat{\mu} + \sum_i \sum_h \hat{\rho}_h + \sum_h \sum_i \hat{a}_{i(h)} + np\hat{\tau}_j + \sum_h \sum_k \hat{b}_{k(h)} + \sum_k \sum_h \hat{(\tau\rho)}_{jh} = y_{.j..} \quad for \ each \ j = 1, ..., p$$

$$\implies \hat{\tau}_j = \bar{y}_{.j..} - \hat{\mu} \quad for \ each \ j = 1, ..., p$$

$$\implies \hat{\tau}_j = \bar{y}_{.j..} - \bar{y}_{....} \quad for \ each \ j = 1, ..., p$$

k-th Column Effect in the h-th Latin Square

$$b_{k(h)} : p\hat{\mu} + \sum_j \sum_i \hat{\rho}_h + \sum_j \sum_i \hat{a}_{i(h)} + \sum_i \sum_j \hat{\tau}_j + p\hat{b}_{k(h)} + \sum_i \sum_j \hat{(\tau\rho)}_{jh} = y_{..kh} \quad for \ each \ k = 1, ..., p \ \ and \ h = 1, ..., n$$

$$\implies \hat{b}_{k(h)} = \bar{y}_{..kh} - \hat{\mu} - \hat{\rho}_h \quad for \ each \ k = 1, ..., p \ \ and \ h = 1, ..., n$$

$$\implies \hat{b}_{k(h)} = \bar{y}_{..kh} - \bar{y}_{...h}, \quad k = 1, .., p \ and \ h = 1, .., n$$

j-th Treatment Interaction Effect in the k-th Latin Square

$$(\tau\rho)_{jh} : p\hat{\mu} + p\hat{\rho}_h + \sum_i \hat{a}_{i(k)} + p\hat{\tau}_j + \sum_k \hat{b}_{k(h)} + p\hat{(\tau\rho)}_{jh} = y_{.j.h}, \quad j = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{(\tau\rho)}_{jh} = \bar{y}_{.j.h} - \hat{\mu} - \hat{\rho}_h - \hat{\tau}_j, \quad j = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{(\tau\rho)}_{jh} = (\bar{y}_{.j.h} - \bar{y}_{....}) - (\bar{y}_{...h} - \bar{y}_{....}) - (\bar{y}_{.j..} - \bar{y}_{....}), \quad j = 1, .., p \ and \ h = 1, .., n$$

$$\implies \hat{(\tau\rho)}_{jh} = \bar{y}_{.j.h} - \bar{y}_{...h} - \bar{y}_{.j..} + \bar{y}_{....}, \quad j = 1, .., p \ and \ h = 1, .., n$$

(b) Reduction in Sum of Squares

Using the solution to the normal equations in (a), the reduction in the sum of squares for the full model is

$$R(\mu, \rho, a, \tau, b, \tau\rho) = \hat{\mu}y_{....} + \sum_{h=1}^{n} \hat{\rho}_h y_{...h} + \sum_{h=1}^{n} \sum_{i=1}^{p} \hat{a}_{i(h)} y_{i..h} + \sum_{j=1}^{p} \hat{\tau}_j y_{.j..} + \sum_{h=1}^{n} \sum_{k=1}^{p} \hat{b}_{k(h)} y_{..kh} + \sum_{h=1}^{n} \sum_{j=1}^{p} \hat{(\tau\rho)}_{jh} y_{.j.h}$$

Therefore, the sum of squares due to ρ_h after fitting $\mu, a_i, \tau_j, b_k, (\tau\rho)_{kh}$ is

$$R(\rho \mid \mu, a, \tau, b, \tau\rho) = R(\mu, \rho, a, \tau, b, \tau\rho) - R(\mu, a, \tau, b, \tau\rho) = \sum_{h=1}^{n} \hat{\rho}_h y_{...h}$$

$$\sum_{h=1}^{n} \hat{\rho}_h y_{...h} = \sum_{h=1}^{n} (\bar{y}_{...h} - \bar{y}_{....}) y_{...h} = \sum_{h=1}^{n} \frac{1}{p^2} y^2_{...h} - \frac{1}{np^2} y^2_{....}$$

Hence,

$$SS_{Latin\ Square} = \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h} - \frac{1}{np^2} y^2_{....}$$

Using the same argument

$$R(a|\mu, \rho, \tau, b, \tau\rho) = \sum_{h=1}^{n} \sum_{i=1}^{p} \hat{a}_{i(h)} y_{i..h} = \sum_{h=1}^{n} \sum_{i=1}^{p} (\bar{y}_{i..h} - \bar{y}_{...h}) y_{i..h} = \frac{1}{p} \sum_{h=1}^{n} \sum_{i=1}^{p} y^2_{i..h} - \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h}$$

Hence,

$$SS_{Row} = \frac{1}{p} \sum_{h=1}^{n} \sum_{i=1}^{p} y^2_{i..h} - \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h}$$

Using symmetry we can obtain the sum of squares due to column effect

$$SS_{Column} = \frac{1}{p} \sum_{h=1}^{n} \sum_{k=1}^{p} y^2_{..kh} - \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h}$$

Using the same argument as previously

$$R(\tau|\mu, \rho, a, b, \tau\rho) = \sum_{j=1}^{p} \hat{\tau}_j y_{.j..} = \sum_{j=1}^{p} (\bar{y}_{.j..} - \bar{y}_{....}) y_{.j..} = \frac{1}{np} \sum_{j=1}^{p} y^2_{.j..} - \frac{1}{np^2} y^2_{....}$$

Hence,

$$SS_{Treatment} = \frac{1}{np} \sum_{j=1}^{p} y^2_{.j..} - \frac{1}{np^2} y^2_{....}$$

Using the same argument

$$R(\tau\rho|\mu, \rho, a, \tau, b) = \sum_{h=1}^{n} \sum_{j=1}^{p} \hat{\tau\rho}_{jh} y_{.j.h} = \sum_{h=1}^{n} \sum_{j=1}^{p} (\bar{y}_{.j.h} - \bar{y}_{...h} - \bar{y}_{.j..} + \bar{y}_{....}) y_{.j.h}$$

$$= \frac{1}{p} \sum_{h=1}^{n} \sum_{j=1}^{p} y^2_{.j.h} - \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h} - \frac{1}{np} \sum_{j=1}^{p} y^2_{.j..} + \frac{1}{np^2} y^2_{....}$$

Therefore,

$$SS_{Interaction} = \frac{1}{p} \sum_{h=1}^{n} \sum_{j=1}^{p} y^2_{.j.h} - \frac{1}{p^2} \sum_{h=1}^{n} y^2_{...h} - \frac{1}{np} \sum_{j=1}^{p} y^2_{.j..} + \frac{1}{np^2} y^2_{....}$$

Furthermore,

$$SS_{Total} = \sum_{i} \sum_{k} \sum_{j(i,k)} \sum_{h} (y_{ijkh} - \bar{y}_{....})^2 = \sum_{i} \sum_{k} \sum_{j(i,k)} \sum_{h} y^2_{ijkh} - \frac{1}{np^2} y^2_{....}$$

Finally, the anova table has the following form

Source	Sum of Squares	Degrees of Freedom
Treatment	$\frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 - \frac{1}{np^2} y_{....}^2$	$p-1$
Row	$\frac{1}{p}\sum_{h=1}^{n}\sum_{i=1}^{p} y_{i..h}^2 - \frac{1}{p^2}\sum_{h=1}^{n} y_{...h}^2$	$n(p-1)$
Column	$\frac{1}{p}\sum_{h=1}^{n}\sum_{k=1}^{p} y_{..kh}^2 - \frac{1}{p^2}\sum_{h=1}^{n} y_{...h}^2$	$n(p-1)$
Latin Square	$\frac{1}{p^2}\sum_{h=1}^{n} y_{...h}^2 - \frac{1}{np^2} y_{....}^2$	$n-1$
Interaction	$\frac{1}{p}\sum_{h=1}^{n}\sum_{j=1}^{p} y_{.j.h}^2 - \frac{1}{p^2}\sum_{h=1}^{n} y_{...h}^2 - \frac{1}{np}\sum_{j=1}^{p} y_{.j..}^2 + \frac{1}{np^2} y_{....}^2$	$(p-1)(n-1)$
Error	SSE	$n(p-1)(p-2)$
Total	$\sum_{i}\sum_{k}\sum_{j(i,k)}\sum_{h} y_{ijkh}^2 - \frac{1}{np^2} y_{....}^2$	np^2-1

where The sum of squares due to error can be computed as

$$SSE = SS_{Total} - SS_{Treatments} - SS_{Row} - SS_{Column} - SS_{Latin\ Square} - SS_{Interaction}$$

and its degrees of freedom are

$$df_{Error} = np^2 - 1 - (p-1) - 2n(p-1) - (n-1) - (n-1)(p-1) = n(p-1)(p+1) - (p-1)3n = n(p-1)(p-2)$$